Game Theoretic Model

for

Portfolio Selection

Comparitive Analysis And Applications To The Indian Stock Market
# Table of Contents

1 Introduction ........................................................................................................... 3
2 Analytical Models .................................................................................................. 3
   2.1 The Mean Variance Model (CAPM) ................................................................. 4
   2.2 The Minimax Model ......................................................................................... 5
3 Data & Procedures ................................................................................................. 6
4 Calculations ........................................................................................................... 6
   4.1 Sample MV Calculation .................................................................................... 6
   4.2 Sample Minimax Calculation ........................................................................... 8
5 Conclusions ........................................................................................................... 9
6 Index .................................................................................................................... 12
7 References ............................................................................................................ 12
8 Appendix .............................................................................................................. 13
   8.1 Game Theory .................................................................................................. 13
      8.1.1 What Is A Game? ...................................................................................... 13
      8.1.2 Representation of Games .......................................................................... 13
      8.1.3 Types of Games ....................................................................................... 14
      8.1.4 Nash Equilibrium ..................................................................................... 15
1 Introduction

The Capital Asset Pricing Model (CAPM) proposed by Treynor (1962) and Sharpe (1964) and refined by many others produces broad results which have many interpretations. CAPM as it stands today doesn’t account for arbitrage. It assumes that there is general equilibrium and marginal utilities for current and future consumption are known. This combined with stochastic models incorporates a rich variety of economic ideas and produce results which are disputed for their empirical validity. This can be attributed to the many different perspectives that CAPM offers on pricing data but importantly the stochastic models which are very ambivalent. These models produce results that are to be correlated to two things, namely behaviour of returns and investor beliefs. These two goals do not necessarily mix. Investors do not have perfect information and are mostly mistaken about future predictions on price. The fact that stochastic models try to fit past data does not indicate any accuracy in future predications.

Game Theory has been applied deeply in analysing previously inexplicable scenarios in finance. Ross (1977) applied Game Theory in the form of a signalling model to produce fairly accurate results in modelling price behaviour from signals from company managers in the form of dividend declarations. In this report, the Minimax rule is used to optimally select a portfolio.

CAPM is based on the Mean Variance (MV) optimization model. The MV model solves the portfolio problem by using two pieces of information: risk and expected returns. Expected returns are represented by mean return of the asset and risk is represented by the variation of the return.

The MV model’s drawbacks led researchers to develop alternative portfolio selection models. The Minimax (Young, 1998) and the Mean Absolute Deviation (MAD) portfolio selection models are now frequently employed in portfolio studies. Both models make use of linear equations, removing one of the MV model’s major shortcomings and making them more suitable for practical use.

The goal of this study is to review the application and relative performances of the Minimax, the MV model and a modified Minimax model.

2 Analytical Models

Portfolio theory (See Markowitz, 1952) is applied to the study of methods for selecting portfolios under risky conditions. Following this theory, a portfolio’s expected return is estimated based on a probability distribution that takes into account the investor’s utility function. A density function of the events is built, and its measure of central tendency gives the return from the assets; correspondingly, its standard deviation, which is the dispersion measure of the
expected returns around the mean, is a convenient measure of asset risk. In the Mean-Variance (MV) model, the portfolio that minimizes the variance subject to the restriction of a given mean return is chosen as the optimum portfolio.

The MAD and Minimax models measure risk in an alternative way. The MAD model retains some of the theoretical characteristics of the MV model, and because of this, it is frequently used. The Minimax model is based on game theory and has also been employed in portfolio optimization studies.

### 2.1 The Mean Variance Model

According to Young (1998), the MV model can be described as:

\[
\min_w \sum_{j=1}^{N} \sum_{k=1}^{N} w_j w_k s_{jk}
\]

Subject to

\[
\sum_{j=1}^{N} w_j \bar{y}_j \geq G
\]

\[
\sum_{j=1}^{N} w_j = W
\]

\[
0 \leq w_j \leq u_j, j = 1 \ldots n \text{ and } k = 1 \ldots n
\]

Where \( s_{jk} = \frac{1}{(T-N)} \sum_{t=1}^{T} (y_{jt} - \bar{y}_j)(\bar{y}_{kt} - \bar{y}_k) \) for a finite number of assets \( N \) at a time \( T \). \( y_{jt} \) denotes the return of the asset \( j \) at time \( t \); \( \bar{y}_j \) is the average return of the asset \( k \); \( \bar{y}_k \) is the average portfolio return; \( w_j \) are the portfolio allocations of the assets, respectively; and \( u_j \) is the maximum budget share allocated to each asset \( j \).

The MV portfolio selection model represents the portfolio with minimum variance (Eq. 1), subject to the restriction that the mean return of the portfolio overcomes a given level, \( G \) (Eq. 2), such that total allocations to the portfolio cannot exceed the total budget, \( W \) (Eq. 3). The significant outcome of this analysis comes from the fact that as the correlation among the assets decreases, the benefits of the portfolio’s diversification increases, that is, the risk level decreases for a given return rate. Thus, the lower the correlation among asset returns, the higher the risk diversification will be. Note that many market agents do not consider the standard deviation of returns as a satisfactory portfolio risk measurement (Kroll et al., 1984; Young, 1998). Also, as investors’ perception of risk may not be symmetrical, the Markowitz model should be seen only as an approximation of the investor’s optimization problem.
2.2 The Minimax Model

Young (1998) was the first to apply the Minimax model for portfolio selection. As mentioned before, this model is based on game theory. A game can have two or more players each one knowing the goals and the opponents’ possible strategies (complete information games). If each player behaves rationally, then game theory asserts that a solution for every situation can be determined by assuming that the players seek to maximize their expected minimum returns – Maximin criterion – or, conversely, minimize their maximum expected losses – Minimax criterion. Situations that involve the agents’ decision-making process under risk conditions have been very well represented and solved through game theory. Even though those situations usually involve only one agent, the Minimax model has shown to be suitable for solving those kinds of problems, as long as Nature is considered the other player and the player who makes the decisions protects himself from the worst possible outcome.

According to Young (1998), the formulation of the Minimax model applied to portfolio selection can be described as follows: For a finite number of financial assets, N, and horizon, T:

\[
\bar{y}_j = \frac{1}{T} \sum_{t=1}^{T} y_{jt} \\
E_p = \sum w_j \bar{y}_j \\
y_{pt} = \sum_{j=1}^{N} w_j y_{jt} \\
M_p = \min_{t} y_{pt}
\]

Where \(y_{jt}\) denotes the return of money invested in asset j at time t; \(\bar{y}_j\) is the mean return from asset j; \(w_j\) is the portfolio allocation to asset j; \(y_{pt}\) is the portfolio return at time t; \(E_p\) is the average portfolio return; and \(M_p\) is the portfolio minimum return from the period.

This formulation refers to the description of the Maximin portfolio selection method; however, the term Minimax will be used since it is more often mentioned in the specialized literature for this formulation. Nevertheless, it is necessary to highlight the fact that the formulation presented here is the Maximin criterion, which is not its dual formulation, Minimax.
The Minimax model attempts to obtain the maximum value of $M_p$ (the portfolio minimum return time period, such that $E_p$ (portfolio mean return) exceeds a certain level, $G$ and total portfolio allocations cannot exceed the total budget $W$.

3 Data & Procedures

The initial step in the selection of the optimum portfolio of NSE stocks during a selected month is the compilation of data from that market for the previous 5 years. In this report, transaction costs and taxes are ignored. The maximum budget share ($u_i$) that could be allocated to a single asset was set at 30% for the Minimax model simulation. There is no theoretical foundation to justify any restriction to the budget allocation share to a single asset. Nevertheless, many authors have suggested those constraints, among them Papahristodoulou (2003).

Through use of the MV and Minimax models, optimal portfolios were generated for three time periods from a pre-selected universe of 4 arbitrarily selected stocks. The stocks available for selection came from the S&P CNX NIFTY Index. This index is composed of 50 stocks. The data was complied from Yahoo! Finance (http://finance.yahoo.com).

All simulations were run on Microsoft Excel using the solver add-in

4 Calculations

The study was done on three blocks of data to simulate one portfolio for each blocks data. Twelve months of preceding data was used to generate weights for optimal portfolios via the MV model and the Minimax model. The periods taken were January 2008 to December 2008 to generate a portfolio for January 2009, January 2009 to December 2009 to generate a portfolio for January 2010 January 2010 to December 2010 to generate a portfolio for January 2011.

Shortselling was allowed in the MV model but not accommodated in the Minimax model.

4.1 Sample MV Calculation

This is a sample calculation for the MV model for the period from January 2008 to December 2008 to generate the weights for a portfolio in January 2009.

Initially, the mean returns were calculated.

<table>
<thead>
<tr>
<th>Asset Returns</th>
<th>ITC</th>
<th>L&amp;T</th>
<th>HDFC</th>
<th>REL</th>
</tr>
</thead>
</table>
Using this data, a covariance matrix was generated using the data analysis tool in Excel.

<table>
<thead>
<tr>
<th></th>
<th>ITC</th>
<th>L&amp;T</th>
<th>HDFC</th>
<th>REL</th>
</tr>
</thead>
<tbody>
<tr>
<td>ITC</td>
<td>0.022133</td>
<td>0.002909</td>
<td>-0.15537</td>
<td>0.006212</td>
</tr>
<tr>
<td>L&amp;T</td>
<td>0.002909</td>
<td>0.005335</td>
<td>0.009996</td>
<td>0.003907</td>
</tr>
<tr>
<td>HDFC</td>
<td>-0.15537</td>
<td>0.009996</td>
<td>1.372743</td>
<td>-0.0252</td>
</tr>
<tr>
<td>REL</td>
<td>0.006212</td>
<td>0.003907</td>
<td>-0.0252</td>
<td>0.007148</td>
</tr>
<tr>
<td><strong>Variances</strong></td>
<td>0.002905</td>
<td>-0.00043</td>
<td>0.000354</td>
<td>0.000691</td>
</tr>
</tbody>
</table>

Table 2 (MV Covariance Matrix)
An initial estimate was made on individual asset weights to generate an estimate of the portfolio variance from the covariance matrix.

<table>
<thead>
<tr>
<th></th>
<th>ITC</th>
<th>L&amp;T</th>
<th>HDFC</th>
<th>REL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Portfolio Allocation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ITC</td>
<td>0.25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L&amp;T</td>
<td>0.25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HDFC</td>
<td>0.25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>REL</td>
<td>0.25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3 (MV Initial Weights)
Using the solver tool in Excel, the portfolio variance was minimised to generate the expected return and calculate the final weights.
Table 4 (MV Final Weights)

<table>
<thead>
<tr>
<th>Portfolio Allocation</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ITC</td>
<td>0.825</td>
</tr>
<tr>
<td>L&amp;T</td>
<td>-0.122</td>
</tr>
<tr>
<td>HDFC</td>
<td>0.100</td>
</tr>
<tr>
<td>REL</td>
<td>0.196</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>1</strong></td>
</tr>
</tbody>
</table>

4.2 Sample Minimax Calculation

This is a sample calculation for the Minimax model for the period from January 2008 to December 2008 to generate the weights for a portfolio in January 2009.

The mean returns for 12 months of stock data were computed.

<table>
<thead>
<tr>
<th>Asset Returns</th>
<th>ITC</th>
<th>L&amp;T</th>
<th>HDFC</th>
<th>REL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.068868</td>
<td>-0.16886</td>
<td>-0.13729</td>
<td>-0.13167</td>
</tr>
<tr>
<td>Mean Returns</td>
<td>-0.018297</td>
<td>0.003323</td>
<td>0.382054</td>
<td>-0.00961</td>
</tr>
</tbody>
</table>

Table 5 (Minimax Mean Returns Calculations)

Standard deviation and variances were calculated for the returns data.

<table>
<thead>
<tr>
<th>Standard Deviation</th>
<th>0.1487721</th>
<th>0.073044</th>
<th>1.171641</th>
<th>0.084543</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variance</td>
<td>0.0221331</td>
<td>0.005335</td>
<td>1.372743</td>
<td>0.007148</td>
</tr>
</tbody>
</table>

Table 6 (Minimax Standard Deviation & Variance)
A table of worst returns was computed by subtracting standard deviations from mean returns.

| Worst Returns | 0.16707 | 0.06972 | 0.78959 | 0.09415 |

Table 7 (Worst Returns Minimax)

Weights were assigned to assets upto the maximum level to the asset with the lowest worst return.

<table>
<thead>
<tr>
<th>Portfolio Allocation</th>
<th>ITC</th>
<th>L&amp;T</th>
<th>HDFC</th>
<th>REL</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>ITC</td>
<td>0.100</td>
<td></td>
<td></td>
<td></td>
<td>1.000</td>
</tr>
<tr>
<td>L&amp;T</td>
<td>0.300</td>
<td></td>
<td></td>
<td></td>
<td>1.000</td>
</tr>
<tr>
<td>HDFC</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
<td>1.000</td>
</tr>
<tr>
<td>REL</td>
<td>0.300</td>
<td></td>
<td></td>
<td></td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 8 (Final Weights Minimax)

5 Conclusions

The results that were computed are enumerated in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical MV Return</td>
<td>0.020983</td>
<td>0.051365</td>
<td>0.083659</td>
</tr>
<tr>
<td>Actual MV Return</td>
<td>0.051537</td>
<td>-0.06048</td>
<td>-0.00909</td>
</tr>
<tr>
<td>Correlation</td>
<td>-0.52567</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theoretical Minimax Return</td>
<td>-0.00372</td>
<td>0.084979</td>
<td>-0.0174</td>
</tr>
<tr>
<td>Actual Minimax Return</td>
<td>-0.00324</td>
<td>-0.00938</td>
<td>-0.03216</td>
</tr>
<tr>
<td>Correlation</td>
<td>0.429807</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Index Return</td>
<td>-0.03136</td>
<td>-0.05805</td>
<td>-0.03866</td>
</tr>
<tr>
<td>Theoretical MV Variance</td>
<td>0.003522</td>
<td>0.051365</td>
<td>0.006262</td>
</tr>
<tr>
<td>Actual MV Variance</td>
<td>0.030554</td>
<td>0.111848</td>
<td>0.092745</td>
</tr>
<tr>
<td>Correlation</td>
<td>0.718235</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theoretical Minimax Variance</td>
<td>0.003465</td>
<td>0.002196</td>
<td>0.002924</td>
</tr>
<tr>
<td>Actual Minimax Variance</td>
<td>0.000478</td>
<td>0.094354</td>
<td>0.014756</td>
</tr>
<tr>
<td>Correlation</td>
<td>-0.95619</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Theoretical MV Variance</td>
<td>0.020383</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Actual MV Variance</td>
<td>0.078382</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Average Theoretical Minimax Variance | 0.002862
---|---
Average Actual Minimax Variance | 0.036529

Theoretical MV over Index | 0.05234 | 0.109416 | 0.122323
---|---|---|---
Actual MV over Index | 0.082894 | -0.00243 | 0.029578
Theoretical Minimax over Index | 0.027641 | 0.143029 | 0.021259
Actual Minimax over Index | 0.02812 | 0.048675 | 0.006503

Table 9 (Results Table)

An interesting result comes into the data in the correlations is that the actual and theoretical MV values are strongly correlated whereas the opposite is true in the case of the Minimax model. Also it can be noted that the variances are correlated for the MV model as opposed to the negative correlation for the Minimax model.
We note that the actual Minimax variance is significantly larger than the theoretical MV variance. The Minimax model thus fails to match the MV model in predicting variances in results.

It is evident from the data that the Minimax theoretical performance over the index's values are closer to the actual values than for the MV model where the theoretical values appear to be an order of magnitude higher than the actual values over the index.

We take the average performance over index as the measure of success in this comparative analysis. We see that the MV model delivers 32% higher returns than the Minimax model. This is a very significant measure of success for the MV model over the Minimax model. However, the variance of the MV model is 115% greater than the variance in the Minimax model. As a fraction of the return, the variance of the MV model is 4.2 times the return on average whereas the Minimax model has the ratio at 3.6. This shows that Minimax model is more accurate in predicting it's outcome.

We conclude therefore that the MV model generates higher returns with greater variability whereas the Minimax model generates lower returns with great accuracy.

A shortcoming of this study is the sparseness of data. To verify this finding a larger dataset must be analysed. Also, the implications of these results on the Efficient Market Hypothesis should be examined closely.
6 Index

Tables
1. MV Returns Calculations
2. MV Covariance Matrix
3. MV Initial Weights
4. MV Final Weights
5. Minimax Means Returns Calculations
6. Minimax Standard Deviation & Variance
7. Minimax Worst Returns
8. Minimax Final Weights

7 References

7.1 Research Papers


7.2 Websites

Yahoo! Finance
8 Appendix

8.1 Game Theory

8.1.1 What Is A Game?
Game theory is the study of strategic decision making. More formally, it is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers." An alternative term suggested "as a more descriptive name for the discipline" is interactive decision theory. The games studied in game theory are well-defined mathematical objects. A game consists of a set of players, a set of moves (or strategies) available to those players, and a specification of payoffs for each combination of strategies. Most cooperative games are presented in the characteristic function form, while the extensive and the normal forms are used to define noncooperative games.

8.1.2 Representation of Games

Extensive Form
The extensive form can be used to formalize games with a time sequencing of moves. Games here are played on trees (as pictured to the left). Here each vertex (or node) represents a point of choice for a player. The player is specified by a number listed by the vertex. The lines out of the vertex represent a possible action for that player. The payoffs are specified at the bottom of the tree. The extensive form can be viewed as a multi-player generalization of a decision tree. The extensive form can also capture simultaneous-move games and games with imperfect information To represent it, either a dotted line connects different vertices to represent them as being part of the same information set

Normal Form
The normal (or strategic form) game is usually represented by a matrix which shows the players, strategies, and pay-offs (see the example to the right). More generally it can be represented by any function that associates a payoff for each player with every possible combination of actions. In the accompanying example there are two players; one chooses the row and the other chooses the column. Each player has two strategies, which are specified by the number of rows and the number of columns. When a game is presented in normal form, it is presumed that each player acts
simultaneously or, at least, without knowing the actions of the other. If players have some information about the choices of other players, the game is usually presented in extensive form. Every extensive-form game has an equivalent normal-form game, however the transformation to normal form may result in an exponential blowup in the size of the representation, making it computationally impractical.

8.1.3 Types of Games

Cooperative or non-cooperative

A game is cooperative if the players are able to form binding commitments. For instance the legal system requires them to adhere to their promises. In noncooperative games this is not possible.

Often it is assumed that communication among players is allowed in cooperative games, but not in noncooperative ones. However, this classification on two binary criteria has been questioned, and sometimes rejected. Of the two types of games, noncooperative games are able to model situations to the finest details, producing accurate results. Cooperative games focus on the game at large. Considerable efforts have been made to link the two approaches. Hybrid games contain cooperative and non-cooperative elements. For instance, coalitions of players are formed in a cooperative game, but these play in a non-cooperative fashion.

Symmetric and asymmetric

A symmetric game is a game where the payoffs for playing a particular strategy depend only on the other strategies employed, not on who is playing them. If the identities of the players can be changed without changing the payoff to the strategies, then a game is symmetric. Many of the commonly studied 2×2 games are symmetric. The standard representations of chicken, the prisoner’s dilemma, and the stag hunt are all symmetric games. Some scholars would consider certain asymmetric games as examples of these games as well. However, the most common payoffs for each of these games are symmetric. Most commonly studied asymmetric games are games where there are not identical strategy sets for both players. For instance, the ultimatum game and similarly the dictator game have different strategies for each player. It is possible, however, for a game to have identical strategies for both players, yet be asymmetric.

Zero-sum and non-zero-sum
Zero-sum games are a special case of constant-sum games, in which choices by players can neither increase nor decrease the available resources. In zero-sum games the total benefit to all players in the game, for every combination of strategies, always adds to zero (more informally, a player benefits only at the equal expense of others). Poker exemplifies a zero-sum game (ignoring the possibility of the house’s cut), because one wins exactly the amount one’s opponents lose. Other zero-sum games include matching pennies and most classical board games including Go and chess. Informally, in non-zero-sum games, a gain by one player does not necessarily correspond with a loss by another. Constant-sum games correspond to activities like theft and gambling, but not to the fundamental economic situation in which there are potential gains from trade. It is possible to transform any game into a (possibly asymmetric) zero-sum game by adding an additional dummy player (often called ”the board”), whose losses compensate the players’ net winnings.

Perfect information and imperfect information

An important subset of sequential games consists of games of perfect information. A game is one of perfect information if all players know the moves previously made by all other players. Thus, only sequential games can be games of perfect information because players in simultaneous games do not know the actions of the other players. Most games studied in game theory are imperfect-information games. Interesting examples of perfect-information games include the ultimatum game and centipede game. Recreational games of perfect information games include chess, go, and mancala. Many card games are games of imperfect information, for instance poker or contract bridge.

Perfect information is often confused with complete information, which is a similar concept. Complete information requires that every player know the strategies and payoffs available to the other players but not necessarily the actions taken. Games of incomplete information can be reduced, however, to games of imperfect information by introducing ”moves by nature”

8.1.4 Nash Equilibrium

In game theory, the Nash equilibrium is a solution concept of a game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his own strategy unilaterally. If each player has chosen a strategy and no player can benefit by changing his
or her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium.